Patterns of Symmetry Breaking in Cosmology and the Laboratory

R. J. Rivers

Theoretical Physics Group; Blackett Laboratory, Imperial College, London SW7 2AZ

In these lectures we look for parallels between symmetry breaking in the early universe and condensed matter systems, and discuss experiments that display these.

I. OVERVIEW

Our understanding of the early universe is strongly hampered by the difficulty in making clean inferences about the nature of its underlying field theory from the observational data. The goal of the COSLAB programme is to find general physical principles common to particle astrophysics (Cosmology) and condensed matter physics (the Laboratory) which permit direct experimental testing in the latter that is impossible in the former.

Phase transitions are ideal in this regard, occurring in both condensed matter systems and the early universe and, in the former case, being directly testable in the laboratory. In these talks I shall examine some of the parallels that phase changes suggest, their consequences and their experimental confirmation. Because of the breadth of these talks, in many cases I have cited review articles, rather than the original references. Brevity requires occasional over-simplification. I apologise to authors who feel unfairly omitted. In particular, I refer the reader to the proceedings of earlier COSLAB meetings [1, 2].

II. PHASE TRANSITIONS

Phase transitions are changes of state, associated with changes of (internal) symmetries of the system. Although they occur in both the early universe and condensed matter systems, their status is different in the two cases:

In the case of the early universe, we have many transitions anticipated on general grounds. However, although we have little knowledge of the symmetries at very high energy scales, we can make some generic predictions. For example, simple models typically produce monopoles and cosmic strings, both of which are observable. Unfortunately our understanding of the non-equilibrium dynamics of the very early universe that would enable us to predict the number of each is very poor (but see Paul Shellard's lectures in these proceedings).

On the other hand, for condensed matter, many transitions are known, for which our detailed knowledge is very good. However, there has been limited experimentation on them in directions parallel to the interests of the cosmologists, which emphasise their non-equilibrium nature. For these reasons, phase transitions in condensed matter provide an ideal testing ground to see to what extent one community can inform the other.

In condensed matter systems familiar changes of state include:

- Non-Magnetic $\rightleftarrows$ Magnetic

- Normal Fluid $\rightleftarrows$ Superfluid

- Normal Conductance $\rightleftarrows$ Superconductance

- Bose-Einstein Condensation

- Liquid crystal formation

There is no hierarchy within these transitions. In particular, they are true transitions, rather than rapid, but smooth, cross-overs. The situation is very different for the early universe, for which early times imply higher energy which, in turn, implies greater symmetry. In consequence, a cooling universe generates a hierarchy of transitions:

- Planck Scale $\qquad\qquad k_B T \sim 10^{19} GeV \ (10^{-45} s)$

- GUT Transitions ** $\qquad k_B T \sim 10^{15} GeV \ (10^{-37} s)$

- Supersymmetry Breaking * $\quad k_B T \sim 10^3 GeV \ (10^{-12} s)$

- Electroweak Transition *** $\quad k_B T \sim 10^2 GeV \ (10^{-10} s)$

- Quark-Hadron Transition **** $k_B T \sim 10^2 MeV \ (10^{-4} s)$

In practice, greater symmetry implies less knowledge (the more stars the better). The 4-star hadronic transition is best known. Its cold phase (the hadrons of accelerator physics) is extremely well understood. The transition is unique among early universe transitions in that the hot phase, the quark-gluon plasma (QGP), can be simulated experimentally, in some sense, by colliding heavy ions at high enough energy. Our theoretical understanding of deconfinement is also good, even if we cannot pin down all the details. However, there are also high baryon number superconductor/superfluid transitions that could be important in neutron stars, which we cannot recreate. As we shall see, the transitions are often cross-overs, but this will turn out to be one of the less important differences.

For the 3-star electroweak transition particle accelerators have given us the main properties of the cold phase (but for the Higgs sector). The hot phase has been estimated theoretically to suggest a complicated transition, again a crossover, but in this case there is no direct experimental confirmation.

For our 2-star GUT transitions there is no direct experimental evidence on either side of the transition tem-

perature(s). Our belief in their likelihood is largely motivated by the observed convergence of coupling strengths at GUT scales due to the screening behaviour of field fluctuations. Our 1-star supersymmetry transition, which balances fermionic against bosonic degrees of freedom, has even less direct support. We are motivated by the solution that supersymmetry brings to the hierarchy problem (the need to keep GUT and electroweak symmmetry breaking scales separate) and the fact that theories that incorporate gravity at a quantum level require supersymmetry. At the Planck scale we know nothing reliably, but it sets the absolute scale against which all others are compared.

III. PATTERNS OF SYMMETRY BREAKING

For true transitions a change of phase is typically identified by the vanishing of an *order parameter* (or of one or more components of an order parameter vector or matrix). We have two cases to consider:

- Bosonic Theories: The order parameter fields are bosonic $\phi_a(x)$ from which we construct order parameters as ground state expectation values $\bar{\phi}_a = \langle \phi_a \rangle$. Typically, the change in symmetry is implemented by symmetry breaking in a scalar (semi)classical field potential.

- Fermionic Theories: Fermionic fields have zero ground state expectation values. The order parameters are constructed from expectations of bosonic fermion bilinears, such as $\langle \bar{\psi}\psi \rangle$ (quark condensate) or $\langle \psi^{\uparrow}\psi^{\downarrow} \rangle$ (Cooper Pair). Often we can write the underlying fermionic theory as an effective bosonic order-parameter field theory, so that we can still use simple scalar potentials to characterise phases.

There are no problems for *global* symmetry transformations, exemplified by $\phi(x) \rightarrow e^{ie\alpha}\phi(x)$, where α is constant in space-time. However, for *local* symmetries, most simply of the form $\phi(x) \rightarrow e^{ie\alpha(x)}\phi(x)$, with space-time dependent $\alpha(x)$, the symmetry requires the existence of gauge fields that undergo their own transformations

For global symmetries the exemplary groups are the Orthogonal Groups $O(N)$, with a vector $\vec{\phi}(x)$ of order parameter fields $\phi_a(x)$, $a = 1, 2, ... N$, transforming under $O(N)$ as $\phi(x) \rightarrow R\phi(x)$. R is a real $N \times N$ orthogonal matrix, with $\tilde{R}R = I$. If $detR = 1$ the symmetry group is $SO(N)$, rather than $O(N)$. The simplest invariant from which to construct the potential is $\bar{\phi}^2$. [We should also consider the additive Groups Z_N, for which $\phi(x) \rightarrow U\phi(x)$; $U^N = I$. The most common of these is Z_2 : $\phi(x) \rightarrow -\phi(x)$ for a single real field.]

For local symmetries the exemplary groups are the Unitary Groups $U(N)$, although some global symmetries are of this form (e.g. our $U(1)$ example above). The order parameter fields most simply transform under the N-dimensional vector representation as $\phi(x) \rightarrow U(x)\phi(x)$,

where $U^{\dagger}U = I$. With the N-component ϕ complex, the basic invariant is $\phi^{\dagger}(x)\phi(x)$. If $detU = 1$ then the group is $SU(N)$. As we noted, the requirement of local symmetry invariance demands the existence of gauge fields $A_{\mu}^a(x)$, which transform under the adjoint representation of the group. If we restrict ourselves to unitary groups $U = U(x)$ there are N^2 gauge fields $A_{b\mu}^a(x)$ $(a, b = 1, 2, ..., N)$ before gauge fixing. For $SU(N)$, with $A_{b\mu}^a(x)$ traceless, there are $N^2 - 1$ gauge fields.

Note that $U(1) = SO(2) : \phi^{\dagger}\phi = \phi_1^2 + \phi_2^2$. Also note that $SU(2) \approx SO(3)$, $SO(3) \times SO(3) \approx SO(4)$, where the $\approx$ means identical behaviour for infinitesimal transformations, but different large-scale behaviour. Other groups $(Sp(N), G_2, E_2, ...)$ are rarely used.

A. General Symmetry Breaking

Transitions occur when the system has degenerate ground states. That is, the symmetry group G of the Hamiltonian, or the Action (Lagrangian), is not the symmetry group H of the ground states of the system. We write $G \rightarrow H$. H is determined by enumerating the ground states ϕ_0. For all $h \in H \subset G$, we have $h\phi_0 = \phi_0$. This enables us to identify the manifold of ground states $\mathcal{M}$ as the left cosets $\mathcal{M} = G/H$.

From our comments above, we shall restrict ourselves to spontaneous symmetry breaking (SSB) visible in a classical, or effective, bosonic potential. However, sometimes the symmetry of the Hamiltonian (and the degeneracy) is only approximate, and we may need additional explicit symmetry breaking by mass terms, etc. Such terms can induce significant quantitative changes. In particular, true transitions are replaced by crossovers. Details on symmetry breaking can be found in many sources, particularly Kibble in [1, 2] and we can only recreate the rudiments here.

B. Some simple examples

The simplest symmetry breaking is that of the Z_2 ($\phi \rightarrow -\phi$) symmetry of the generalised Ising model of a single real field ϕ, with double-well potential $V(\phi) = \frac{1}{4}\lambda(\phi^2 - \eta^2)^2$, and action

$$S = \int d^4x \left[\frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - \frac{\lambda}{4}(\phi^2 - \eta^2)^2 \right], \qquad (1)$$

or free energy

$$F = \int d^3x \left[\frac{1}{2}(\nabla\phi)^2 + \frac{\lambda}{4}(\phi^2 - \eta^2)^2 \right]. \qquad (2)$$

The maximum of V at $\phi = 0$ is unstable, but on expanding about the minimum $\phi = \eta$, as $\phi = \eta + h$, we see that $V(\phi) = \frac{1}{2}m_h^2 h^2 + (non\text{-}quadratic \; terms)$ describes h-field excitations of mass $m_h = \eta\sqrt{2\lambda}$. Expanding about

$\phi = -\eta$ gives an identical spectrum. Z_2 symmetries have a natural role in Grand Unified Theories (GUTs) of the early universe, where they provide a simple selection rule against the rapid decay of the proton.

The next model, to which we shall return repeatedly, is the Goldstone model of a complex field $\phi = \phi_1 + i\phi_2$, with Mexican-hat/wine-bottle potential $V(\phi) = \frac{\lambda}{4}(|\phi|^2 - \eta^2)^2$ (and kinetic/gradient terms changed correspondingly in (1) and (2)). The global $O(2)$ symmetry of rotations in the complex ϕ plane is totally broken by the inequivalent, but equally acceptable, degenerate ground states with $|\phi| = \eta$, which form the manifold $\mathcal{M} = S^1$ (the 1-sphere, or circle). Expanding about $\phi = \eta$ say, shows that the particle spectrum is that of a massive (Higgs) particle, mass m_h as above, corresponding to radial oscillations in the complex ϕ-plane, and a massless (gapless) mode, the Goldstone particle, corresponding to translations of the field along S^1.

The natural generalisation of the Goldstone model is to N scalar fields ϕ_a, $a = 1, 2, ..., N$, with potential $V = \frac{\lambda}{4}(\vec{\phi}^2 - \eta^2)^2 = \frac{\lambda}{4}(\phi_a^2 - \eta^2)^2$, where $\phi_a^2 = \phi_1^2 + \phi_2^2 + ...$. The $O(N)$ symmetry is broken as $O(N) \rightarrow O(N-1)$, with a manifold of groundstates $\vec{\phi}^2 = \eta^2$, the $N-1$ dimensional sphere $\mathcal{M} = S^{N-1}$. If we expand about $\phi_1 = \eta, \phi_2 = \phi_3 = ... = 0$, the oscillations describe 1 massive (Higgs) field in the radial field direction and $N-1$ massless (Goldstone) fields along the surface of the sphere of groundstates. Such a model is termed the $O(N)$ Linear Sigma Model ($L\sigma M$). The $O(N)$ Non-Linear Sigma Model ($NL\sigma M$) is an extreme version of this, describing classical rotors, in which, by fixing $\phi_a^2 = \eta^2$, we eliminate the Higgs field as a degree of freedom. There are no known Goldstone modes in particle physics, although there are very light degrees of freedom (e.g. pions), but Goldstone modes arise naturally in condensed matter physics as phonons.

For gauge theories it is not sufficient to just examine the potential $V(\phi)$, because the local nature of the gauge transformations enforces additional interactions through the covariant derivatives that extend the kinetic terms. The simplest case is the $U(1) \rightarrow 1$ Abelian Higgs model for complex field ϕ and gauge field A_μ. The action is

$$S = \int d^D x \left[|D_\mu \phi|^2 - \frac{\lambda}{4}(|\phi|^2 - \eta^2)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right],$$

where $D_\mu = \frac{\partial}{\partial x_\mu} - ieA_\mu$ is the covariant derivative, and $F_{\mu\nu}$ the electromagnetic field tensor. Although $\mathcal{M} = S^1$, as for the Goldstone model, there are no Goldstone modes. The rearrangement of the field degrees of freedom gives rise to one real Higgs particle, and one *massive* vector field. In some sense the Goldstone mode is swapped for the transverse degree of freedom of the vector field.

IV. SYMMETRY BREAKING IN CONDENSED MATTER

The simple forms of SSB given above are most easily realised in condensed matter theory. We shall largely restrict ourselves to continuous transitions for which relevant experiments have been performed.

A. Superfluid ^4He:

This is the global $O(2)$ Goldstone model already discussed, with complex order parameter field $\phi = \rho e^{i\theta}$. In the 2-fluid model $n_s = \rho^2$ is the superfluid density and $\mathbf{v}_s = \frac{\hbar}{m}\nabla\theta$ the superfluid velocity. The Goldstone mode describes sound. As before, $\mathcal{M} = S^1$.

B. Low -T_c Superconductors

The bosonic effective order parameter field ϕ is derived from the $L = S = 0$ Cooper pairs of electrons (with momenta close to the Fermi surface) as $\phi \sim \langle \psi_\downarrow \psi_\uparrow \rangle_{L=S=0}$. This is an Abelian Higgs model with Free Energy

$$F = \int d^3 x \left[\frac{\hbar^2}{2m^*} \left| \left(-i\nabla - \frac{e^*}{\hbar c}\mathbf{A} \right)\phi \right|^2 + \lambda(|\phi|^2 - \eta^2)^2 \right],$$
(3)

in the absence of external fields, where $e^* = 2e$, $m^* = 2m_e$ characterise the Cooper pair. The Meissner effect, whereby the magnetic field penetrates into the bulk superconductor, has its penetration length determined by the mass of the vector field in the broken phase.

C. High-T_c Superconductors

High-T_c superconductivity is a complicated phenomenon. For our purposes, we adopt an idealised explicitly broken $SO(5)$ model in $D = 2$ dimensions [3]. The basic idea is that doping an antiferromagnet leads to d-wave superconductivity. The bosonic effective order parameters are, again, constructed from fermionic bilinears.

We begin with a global $O(3)_{AF}$ LσM for antiferromagnetism with order parameter field (staggered magnetism) $\vec{n}$ with potential $V(\vec{n}) = \frac{\lambda}{4}(\vec{n}^2 - 1)^2$. This is extended to a five-component order parameter field $\vec{N} = (\phi_1, n_1, n_2, n_3, \phi_2)$, with $O(5)$-invariant potential $V(\vec{N}) = \frac{\lambda}{4}(\vec{N}^2 - 1)^2$. [We can work equivalently [3, 4] with a NLσM, in which $|\vec{N}| = 1$.] Ultimately we shall couple $\phi = (\phi_1 + i\phi_2)$ locally to the EM field as in Low-T_c superconductors.

In the first instance $\mathcal{M} = S^4$. We break the $O(5)$ invariance *explicitly* to $O(3)_{AF} \times SO(2)$ by the addition of antiferromagnet and doping terms to the potential,

most simply as [5]

$$V(\phi, \vec{\eta}) = \frac{1}{2} a_S |\phi|^2 + \frac{1}{2} a_A \vec{n}^2 + \frac{b}{4}(|\phi|^2 + \vec{n}^2)^2, \qquad (4)$$

where $a_S, a_R < 0$. We then couple ϕ to EM as

$$F = \int d^2 x \left[\frac{\hbar^2}{2m^*} \left| \left(-i\nabla - \frac{e^*}{\hbar c} \mathbf{A} \right) \phi \right|^2 + \frac{\hbar^2}{2m^*} (\nabla \vec{n})^2 + V(\phi, \vec{n}) \right]. \qquad (5)$$

Increasing the doping drives $|a_S| > |a_R|$, making the $U(1) \sim SO(2)$ superconducting direction the global minimum. That is, $O(5) \to O(3)_{AF} \times SO(2) \to SO(2)$, with $\mathcal{M} = S^1$. This $U(1) \sim SO(2)$ is then broken as for the low-temperature superconductors.

D. Superfluid ^3He

^3He is a Fermi Liquid, which can become superfluid by the formation of p-wave (L=S=1) 'Cooper pairs' of ^3He atoms. L and S are uncoupled at short distances, to give a global symmetry group $G = SO(3)_L \times SO(3)_S \times U(1)_N$.

The effective order parameters [6] form a 3×3 matrix $A_{\alpha i}(x)$, formed from the Fermi bilinears $\langle \psi(x)\psi(x) \rangle_{L=S=1}$. The label $i = 1, 2, 3$ is the orbital angular momentum label and $a = 1, 2, 3$ the spin label. The $U(1)_N$ describes the overall phase freedom. Above the transition all the elements of the matrix have zero values. Below the transition, some of these quantities become non-zero. The symmetry of the order parameter after the transition corresponds to the manifold of symmetries which remain unbroken.

The free energy of these states can be expressed in the framework of the phenomenological Ginzburg-Landau theory by a potential [7]

$$\begin{aligned} V_{GL}(A) = & -\alpha A_{a,i}^* A_{a,i} + \beta_1 A_{a,i}^* A_{a,i}^* A_{b,j} A_{b,j} + \\ & + \beta_2 A_{a,i}^* A_{a,i} A_{b,j}^* A_{b,j} + \beta_3 A_{a,i}^* A_{b,i}^* A_{a,j} A_{b,j} \\ & + \beta_4 A_{a,i}^* A_{b,i} A_{b,j}^* A_{a,j} + \beta_5 A_{a,i}^* A_{b,i} A_{b,j} A_{a,j}^* (6) \end{aligned}$$

The different possible symmetries of the order parameter A_{ai} are identified with local minima and saddle points in this 18-dimensional energy surface. There are two stable phases;

- The A phase, in which

$$SO(3)_S \times SO(3)_L \times U(1)_N \to SO(2)_{S_z} \times U(1)_{L_z - N/2} \times Z_2.$$

  The manifold of ground states is $\mathcal{M}_A = G/H_A = S^2 \times SO(3)/Z_2$. The order parameter in the A phase ground state is anisotropic in both spin and orbital spaces (the 'axial' state). : most simply, it takes the form $A_{a,i}^0 = \Delta_A \hat{z}_a (\hat{x}_i + i\hat{y}_i)$, where $\hat{x}, \hat{y}, \hat{z}$ are unit vectors in the x, y, z directions respectively.

- The B phase, in which the orbital and spin angular momenta are locked together as

$$SO(3)_S \times SO(3)_L \times U(1)_N \to SO(3)_{S+L}.$$

The manifold of ground states is now $\mathcal{M}_B = G/H_B = S^1 \times SO(3)$. As a result, $A_{a,i}$ resembles a rotation matrix. Specifically, in the bulk B phase, $A_{a,i}$ reduces to the arbitrary orthogonal rotation matrix $R_{a,i}$, $A_{a,i} = \Delta R_{a,i} e^{i\Phi}$.

The energy balance between the A and B phases is determined by the relation between the parameters β_i. At zero pressure, the B phase corresponds to the absolute minimum, while at pressures above 20 bar there is a temperature range in which the A phase is preferred.

E. Other systems

There are other systems whose transitions are understood well, which potentially have parallels with the early universe. In particular, we would cite

- Uniaxal nematic liquid crystals, for which the order parameter in the nematic phase is the non-oriented director vector with ground state manifold $RP^2 = S^2/Z_2$. The transition is first order [8]. However, when considering symmetry breaking at the interface of an isotropic-nematic transition the anchoring of the director at the interface forces it to lie on a cone [9], whereby the order parameter space is a circle S^1, corresponding to the familiar $U(1)$ breaking.

- Bose-Einstein condensates, which allow for a great variety of symmetry breaking if species of atoms are mixed. For example, consider two-species BEC (two different ultracold atomic gases in a trap) with order parameter fields $\phi_a(x)$, $(a = 1, 2)$. The trap-independent part of the potential can be written as [10]

$$V = \lambda_1 (|\phi_1|^2 - \eta_1^2)^2 + \lambda_2 (|\phi_2|^2 - \eta_2^2)^2 + \beta |\phi_1|^2 |\phi_2|^2, \quad (7)$$

  with tunable parameters, which is no more than (4) rewritten for $SO(4)$. For a single species (1st term) we have an $O(2)$ symmetry that is totally broken, as in ^4He, and for two species an $O(2) \times O(2)$ symmetry, broken to $O(2)$, with similarity to the breaking of the residual symmetry in high-T_c superconductors.

V. SYMMETRY BREAKING IN THE EARLY UNIVERSE

In general, the overall patterns of symmetry breaking in the field theories that we believe describe the early universe are more complicated than those in condensed matter. In particular, the transitions about which we are most knowledgable (later in time) tend to be crossovers, and the ones we know least are likely to be complicated by virtue of their belonging to larger symmetry groups.

A. The Quark-Gluon System of Quantum Chromodynamics (QCD):

Most known particles are strongly interacting (hadrons), of which the proton, neutron and pion are the lightest members. They are built from quarks (fermions) and gluons (vector bosons). Quarks are described by fields $q_{f,c}$, carrying two labels. The label $c = 1, 2, 3$ is that of a local $SU(3)_c$ symmetry, termed *colour*. In ordinary hadrons the gauge field gluons $A^a_{\mu c}$ ($a, c = 1, 2, 3$), demanded by the local symmetry, form a massless octet ($8 = 3^2 - 1$). The fact that there are no free quarks in hadronic matter at zero chemical potential and temperature (particle accelerators) is explained if all known hadrons are colour *singlets*. The other label $f = 1, 2, .., N$ describes a global $SU(N_f)$ flavour symmetry that is intrinsically broken. At $N_f = 2$ this breaking is very small (electromagnetic), at $N_f = 3$ it is still fairly small. For $3 < N_f \leq 6$ the breaking is large. The resulting theory of quarks and gluons is known as Quantum Chromodynamics.

The transition from a quark-gluon plasma to hadrons at low chemical potential (the early universe and heavy-ion collisions) is complicated by having two different aspects. The first is the approximate breaking of chiral symmetry that is a consequence of the lightness of the common quarks, for which the gluons play a subsidiary role. The second is the confining/deconfining nature of the gauge sector, for which the quarks are relatively unimportant. Although QCD can be tackled directly in some circumstances (through lattice gauge theory), it is informative to think of it as a modification of one of these extremes. More details can be found in the review article by Rajagopal [11].

- **Low energy Chiral theory:** Common hadrons are built from two quarks (termed *up* and *down*) which are approximately massless. One important idealisation is to take them massless in the first instance, $m_d = m_u = 0$, to give a theory that is invariant under independent transformations of the left and right hand components of the quark doublet $\psi = \psi_L + \psi_R$. Writing these as $\psi_L \rightarrow L\psi_L$ and $\psi_R \rightarrow R\psi_R$, where $L, R \in SU(2)_f$ we would have degenerate parity doublets if the $SU(2)_L \times SU(2)_R$ chiral symmetry remains unbroken. This is not the case experimentally. Breaking the symmetry as $SU(2)_L \times SU(2)_R \rightarrow SU(2)_{L+R}$ gives pions as the necessary Goldstone bosons. The pions are not massless (there are no known Goldstone bosons in particle physics) but they are anomalously light, requiring explicit (but small) symmetry breaking. This makes the transition a crossover. In fact, there is an additional $U(1)_A$ that is broken by instantons, without Goldstone bosons, that we shall not consider here. Extensions to $N_f = 3$ (requiring 'strange' quarks with $m_s = 0$) with a more badly broken $SU(3)_L \times SU(3)_R \rightarrow SU(3)_{L+R}$ will

be made later.

- **Effective $O(4)$ sigma models**: Still in the context of chiral theory, the chiral order parameter, a 2×2 matrix M in flavour space, can be written in terms of a pion triplet $\vec{\pi}$, with $\pi^+ = \pi_1 + i\pi_2$, etc, and a singlet σ, as $\langle \bar{\psi}^a_{L\alpha} \psi^{\alpha b}_R \rangle = M^{ab} = \sigma \delta^{ab} + \vec{\pi}.(\vec{\tau})^{ab}$. Low-energy pion physics is well represented by an effective $O(4)$ $L\sigma M$, with potential $V = \lambda(\vec{\pi}^2 + \sigma^2 - f^2)^2$ (or an $O(4)$ $NL\sigma M$ with the constraint $\vec{\pi}^2 + \sigma^2 = f^2$) and explicit term linear in σ to enforce $m_\pi \neq 0$ mass-breaking. In the absence of this, with manifold $\mathcal{M} = S^4$ of ground states the pions are the Goldstone modes (and the Higgs is the σ). Note that $O(4) \rightarrow O(3) \approx O(3) \times O(3) \rightarrow O(3) \approx SU(2) \times SU(2) \rightarrow SU(2)$.

- **Deconfinement** If we take quark masses to be infinitely large, we recover a pure gauge theory. In this case we can construct a non-local order parameter (the Polyakov loop), whose Z_3 symmetry-breaking characterises deconfinement [12]. This is complicated by finite quark masses and is too subtle for us here.

We should stress that there are many other ways to tackle the transitions, but they are even less relevant to our subsequent discussion.

- **Colour Superconductivity:** At high baryon density (chemical potential) there is the possibility of Colour Superconductivity/Superfluidity. This cannot be created in the laboratory but possibly exists in neutron stars, which could have quark matter cores. In cold dense quark matter the relevant degrees of freedom are those of quarks with momenta near to the Fermi surface. Quark attractions will lead to the creation of Cooper pairs. If we begin by assuming unbroken $SU(3)_f$, with massless quarks $m_u = m_d = m_s = 0$, Cooper pairs cannot be flavour singlets, and both colour and flavour symmetry is broken. The unbroken symmetry is (where L and R are chiral flavour)

$$G = SU(3)_L \times SU(3)_R \times SU(3)_c \times U(1)_B,$$

where $U(1)_B$ counts baryons. The possibility exists of colour-flavour locking (CFL), by condensates $\langle \psi^{\alpha a}_L \psi^{\beta b}_L \rangle_{\bar{3}} \propto \Delta \epsilon^{\alpha \beta \gamma} \epsilon^{ab\gamma}$, where α, a denote colour and flavour respectively. This assumes that the $\bar{3}$ channel in $3 \times 3 = 6 + \bar{3}$ is the attractive channel.

$$SU(3)_L \times SU(3)_R \times SU(3)_c \times U(1)_B \rightarrow$$
$$\rightarrow SU(3)_{L+R+c} = SU(3)_{f+c}. \tag{8}$$

Order parameters form a 3×3 complex matrix Σ^a_b built from quark condensates $\langle \psi_L \psi_L \rangle_{\bar{3}}$ and $\langle \psi_R \psi_R \rangle_{\bar{3}}$, as above, expressible as the nine Goldstone bosons arising from the Chiral symmetry

breaking. There is an effective LσM realisation of this that we shall not pursue.

Explicit symmetry breaking from mass terms is necessary. There are several possible phases as we further break the symmetry by breaking the $SU(3)_f$ quark mass degeneracy. On introducing a relatively massive strange quark, $m_s \neq m_u = m_d$, the K^0 is the pseudo-Goldstone boson, and readily forms a condensate. We have what is known as CFL $+K^0$, in which $SU(3)_{c+f} \rightarrow SU(2)_{I'} \times U(1)_{Y'} \rightarrow U(1)$.

The simplest scenario [13] has a doublet Φ of K^0, K^+ charged bosons $\Phi = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}$ for which

$$V(\Phi) = \lambda(|\Phi|^2 - \eta^2)^2 \qquad (9)$$

and $\mathcal{M} = S^3$.

If we include EM effects so that $m_s \neq m_u \neq m_d$ we have the further breaking $SU(3)_{c+f} \rightarrow U(1) \times U(1) \rightarrow U(1)$. The effect of the explicit mass breaking terms is to recreate the potential V of (7) for $(\phi_1, \phi_2) = (K^+, K^0)$.

B. Electroweak Transition/ Standard model:

For systems of low baryon density, we have the unambiguous pattern of symmetry breaking of the Salam-Weinberg model,

$$SU(3)_c \times SU(2)_I \times U(1)_Y \rightarrow SU(3)_c \times U(1)_Q,$$

where $Q = I_3 + \frac{1}{2}Y$ is the electromagnetic charge. This is replicated for each of the three families of quarks and leptons (corresponding to $N_f = (1,2), (3,4), (5,6)$) respectively. Considering only the leptonic sector, the simplest scenario has a doublet Φ of charged bosons

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \qquad \text{for which } V(\Phi) = \lambda(|\Phi|^2 - \eta^2)^2$$

of (9), after field relabelling. However, in this case $V(\Phi)$ is exact, with no further explicit symmetry breaking.

On symmetry breaking we have the familiar three massive vectors ($W^\pm, Z^0$) and one massless gauge field (the photon). The spin-1 nature of the massive vectors has been bought at the expense of three of the field degrees of freedom of the Φ doublet, to leave one massive real Higgs boson. For the assumed Higgs mass ($m_h > 80 GeV$) the phase diagram shows a crossover.

C. Grand Unified Theories:

Ignoring supersymmetry, we assume the existence of symmetries G such that

$$G \rightarrow G_{SM} = SU(3)_c \times SU(2)_I \times U(1)_Y \rightarrow SU(3)_c \times U(1)_Q.$$

e.g $E_6 \rightarrow SO(10) \rightarrow SU(5) \rightarrow SU(3)_c \times SU(2)_I \times U(1)_Y$. In practice, we need an additional discrete symmetry to prevent too rapid a decay of the proton into leptons that is otherwise permitted by standard GUT symmetries. Groups like $SO(10)$ contain a $U(1)_{B-L}$ gauge symmetry, which breaks to give an extended symmetry $G_{SM} \times Z_2$, forbidding fast proton decay. Motivated by the work of Witten [14] and others on superconducting strings , groups G containing $U(1) \times U(1)$, with potentials (4) and (7) have been found useful, for reasons that we shall see later.

However, once we attempt to accommodate supersymmetry at both a minimal level and beyond, the number of possibilities proliferates [15].

VI. SIGNATURES OF TRANSITIONS

The similarity between transitions in condensed matter and the early universe is interesting, but for it to be compelling we need a means of monitoring how transitions take place in condensed matter, so that we can check if similar assumptions about the early universe lead to our understanding it better. The way we do this is by looking for signatures of transitions from which we can infer the nature of the symmetry breaking.

Consider the simplest case of a double-well potential for a real scalar field $\phi(x)$ We rewrite the potential $V(\phi) = \frac{\lambda}{4}(\phi^2 - \eta^2)^2$ as

$$V(\phi) = \frac{1}{2}\phi^2 m^2 + \frac{\lambda}{4}\phi^4 + constant.$$

$m^2 = -\lambda\eta^2$ is negative. All this is at zero temperature. As temperature increases and the field becomes a plasma m^2 becomes temperature dependent, increasing towards zero as $m^2(T) = m^2(1 - T^2/T_c^2)$ (in the simplest Ginzburg-Landau, or one-loop, approximation). The order parameter $\bar\phi = \langle\langle\phi(x)\rangle\rangle$, where the double average is both thermodynamic and quantum mechanical, satisfies $\bar\phi^2 = \eta^2(T) = -m^2(T)/\lambda$. At the critical temperature T_c both $\bar\phi$ and $m^2(T)$ vanish. In relativistic quantum field theory $T_c \sim \eta$, the symmetry breaking scale. For $T > T_c$, $m^2(T) > 0$ and the Z_2 symmetry is restored.

Let us now reverse the order, as in the early universe, and cool the field through T_c from above, breaking the Z_2 symmetry. How is it that symmetry is broken, given the $\phi \leftrightarrow -\phi$ symmetry of the potential? The standard way, in field theory textbooks, is to bias the potential by introducing a spatially uniform external source j coupled to the field as $j\phi$. For $j \neq 0$ there is now a unique ground state to which the system relaxes. On taking $j \rightarrow \pm 0$ the system stays in this groundstate, with $\bar\phi = \pm\eta$ according as the source is removed.

In the early universe there is no such uniform bias. Instead, as we pass through T_c there are *local* thermal fluctuations of the field that will drive it from the unstable maximum at $\phi = 0$ towards one or other of the minima. That is, for some $\vec{x}$, $\phi(\vec{x})$ is driven to $+\eta$, for

some it is driven to $-\eta$, and domains form in which ϕ is correlated. In fact, since the transition is implemented in a finite time, causality (the finite speed at which the field can order itself) requires this domain structure. Once the system is behaving classically, the boundaries of these domains are 'walls', in which the field passes from $\phi = -\eta$ to $\phi = \eta$ according to the classical equation $\delta S / \delta \phi = 0$. Such a 'domain wall' in the $x - y$ plane, with profile $\phi(z) = \eta \tanh(|m|z/\sqrt{2})$, has thickness $\xi_0 \sim |m|^{-1}$ and energy per unit area (surface tension) $\sigma \sim |m|\eta^2$. We will know that the transition has taken place by the existence of these walls. Already we have the powerful result that, if we do not adopt an inflationary scenario to dilute wall formation, then domain walls produced at the GUT scale are so energetic that they would close the universe, in contradiction to the evidence. Although a closed domain wall can contract and disappear, it is not possible to remove part of a wall, and a single wall requires an infinite field reordering to remove. Domain walls are the simplest entities to show this topological stability, and they provide the simplest example of a *topological defect.*

More generally, as we shall see, topological defects are twists, or knots, or other deformation of the fields that carry a topological charge that prevents them unwinding, or untwisting (but not annihilating). Whether defects exist depends on the detailed nature of the symmetry breaking. If they do exist, individual defects produced in the early universe can survive into the present era as 'fossils'. Observing them would provide concrete evidence for transitions having taken place and, if they could be studied, provide information on the symmetry breaking. We can check this out directly with condensed matter systems that permit defects. Further, we can make predictions for defect formation that can be checked in condensed matter, and which have implications for the early universe.

Of course, for different patterns of symmetry breaking the frustration in the order parameter fields as they link domains formed after the transition may not be resolved with topological defects. In such cases the twists in the fields can unwind at no great energy cost and, at times well after the transition, there is no direct signal for it having taken place. In practice, models for the early universe almost inevitably produce defects of one form or another, and the problem is more one of not seeing them.

VII. TOPOLOGICAL DEFECTS

A. Vortices and Strings

The most common topological defect in condensed matter physics and the early universe is the string, or vortex, seen most simply in the breaking of a $U(1)$ symmetry.

Consider a global theory of a complex scalar field $\phi(x) = \rho(x)e^{i\theta(x)}$, with effective potential $V(\phi, T) = \frac{\lambda}{4}(|\phi|^2 - \eta^2(T))^2$, where $\eta(T)$, vanishing at the critical temperature T_c, is given above. As we cool through T_c the field, originally concentrated around the (then) stable minimum at $\phi = 0$, begins to explore the possible groundstates with $\rho = \eta$. With no bias to make ϕ fluctuate away from the unstable maximum at $\phi = 0$ with any particular phase, $\theta(\vec{x})$ will vary from point to point, subject to continuity.

Let us take a closed path $\mathcal{C}$ in space. Once the transition has been completed, $|\phi(x)| \approx \eta$ for $x \in \mathcal{C}$. If we make a complete circuit in $\mathcal{C}$ then $\phi(x)$ executes a closed path in S^1, the circle of ground states. Continuity of $\phi(x)$ requires that the change $\Delta\theta$ in $\theta(x)$ along the path is an integer multiple of 2π, $\Delta\theta = 2\pi n$. We can find classical solutions of infinite extension along the z-axis of the form $\phi_n(\vec{x}) = \rho(r)e^{in\theta}$, where $r^2 = x^2 + y^2$, and $\theta = \tan^{-1}(x/y)$ is the azimuthal angle, that demonstrate this. In ^4He they are vortices with quantised vorticity, of winding number n. It happens that, if $n > 1$, it is energetically advantageous for the vortex to break into n vortices of winding number unity. These are the stable topological defects of the model, being unbreakable. If we shrink the loop $\mathcal{C}$ to the origin $r = 0$, we must have $\rho_n(0) = 0$ for $\phi(x)$ to be defined. That is, the core of the vortex is the false groundstate, just as the core of domain wall was. In low-T_c superconductors the coupling of the electromagnetic and Cooper pair fields makes them Abrikosov vortices with quantised magnetic flex. As happens for ^4He, the Cooper pair field $\phi = 0$ at the vortex core. The thickness of the false vacuum ($\phi \approx 0$) core is the London length, the thickness of the flux tube the Meissner length. In the early universe, with no Goldstone bosons, in general vortices will be of the Abrikosov type.

Suppose we have collection of vortices of the type above. Consider two closed paths $\mathcal{C}_1$ and $\mathcal{C}_2$ with a point in common, in which the field takes winding numbers n_1 and n_2. We can trivially combine them into a closed path $\mathcal{C}_1 + \mathcal{C}_2$ with winding number $n_1 + n_2$. That is, the mappings from closed paths $\mathcal{C}_i$, which we can take to be circles S^1 into the set of ground states $\mathcal{M} = S^1$ form the additive group of integers Z. The group Z is termed the *Fundamental Group* or the *First Homotopy Group* of S^1, written $\Pi_1(S^1)$.

More generally, a transition will produce topologically stable vortices if $\Pi_1(\mathcal{M}) = \Pi_1(G/H)$ is non-trivial. For the condensed matter systems that we have discussed above, we also find topological Z vortices in single-species BEC and liquid crystals at the interface of an isotropic-nematic transition. For our broken $SO(5)$ high-T_c superconductor the situation is more complex, although vortices have winding number $n \in Z$. As we get close to critical doping, and $SO(5)$ symmetry is restored (prior to coupling to electromagnetism) the stable vortices have non-trivial antiferromagnetic cores in which the order parameter $\vec{\eta} \neq \vec{0}$, although $|\phi|$ necessarily vanishes there. As we achieve critical doping the antiferromagnetic core expands to destroy the vortex, since $\Pi_1(S^4)$ is trivial, not

permitting topological charge.

For superfluid ^3He the situation is even more complicated. In the A-phase, $\Pi_1(S^2 \times SO(3)/Z_2) = Z_4$, addition modulo 4. There are four types of vortices, with winding numbers $n = 0, \pm 1/2, \pm 1$ (mod 2). Depending on the parameter values, it is again possible to have vortices with non-trivial cores. In the B-phase, $\Pi_1(S^1 \times SO(3)) = Z \times Z_2$, and there are two types of vortices with arbitrary winding number. Not all vortices are normal (with all components of $A_{\alpha,i}$ vanishing at their cores). It is possible for B-phase vortices to have A-phase cores.

For cosmology, topological strings (vortices) appear in colour superconductivity and superfluidity. In supersymmetric GUT transitions which solve the monopole problem, lead to baryogenesis after inflation and constrain proton decay, the vast majority of models lead to topological (cosmic) strings [15].

We should not ignore so-called *embedded* strings, in which, when $G \to H$, there are subgroups G', H' of G, H respectively, for which $\Pi_1(G'/H')$ is not trivial. Such strings occur in the $O(4)$ sigma model of low-energy pions and in electroweak breaking $SU(2) \times U(1) \to U(1)$. However, such strings have no topological winding number and whether they are energetically classically unstable or not depends upon the values of the parameters (masses, coupling constants). Even if they are the role of quantum fluctuations is not clear. In the important case of electroweak breaking the strings are unstable to begin with, and can only have transitory effects.

Loops of simple relativistic strings contract and turn their energy into particle production. Although vortex loops in ^4He can be stabilised by Magnus forces, in general loops conract. However, many theories (most simply of the $U(1) \times U(1) \to U(1)$ type) can produce strings or vortices that have non-trivial cores with angular momentum, which can stabilize loops (vortons) [16]. Such a mechanism is not possible in ^3He on energetic grounds, although it is possible in 2-component BEC, where it is possible that an example has been seen. The main case of theoretical interest is that of CFL+K^0 quark matter, because of its relevance to neutron stars.

B. Monopoles and more

Consider an $SO(3)$ sigma model with 3-vector field $\vec{\phi}(x)$ and potential $V(\vec{\phi}) = \lambda/4(\vec{\phi}^2 - \eta^2(T))^2$ as it cools through its critical temperature at which $\eta(T_c) = 0$ in a finite time. Causality requires the direction of $\vec{\phi}$ to be uncorrelated at large distances. This frustration is resolved by classical 'knots' in the field, of the form $\vec{\phi}(x) = \rho(r)\vec{x}/r$, where r is the radial distance from the centre of the knot. These are the topological monopoles of the model and cannot be 'unwound'. If we enclose one by a sphere S^2, we see that they are non-trivial mappings of S^2 onto the manifold of ground states $\mathcal{M} = S^2$. These mappings again form a group, the *Second Ho-*

motopy group $\Pi_2(\mathcal{M})$. In fact, $\Pi_2(S^2) = Z$, and the monopole solutions are characterised by integer winding number, for which our example ($|n| = 1$) is stable.

More generally, we have monopoles whenever $\Pi_2(\mathcal{M})$ is non-trivial. In condensed matter physics we find monopoles in the A-phase of superfluid ^3He, for which $\Pi_2(S^2 \times SO(3)/Z_2) = Z$.

Monopoles arise inevitably in GUT transitions. This is a consequence of the fact that, if G is connected and semisimple (i.e. $\Pi_1(G) = 1$), then $\Pi_n(G/H) = \Pi_{n-1}(H)$. Thus, if $\Pi_1(H) \neq 1$, we have monopoles. Since H necessarily contains $U(1)_Q$, $\Pi_1(H) \neq 1$. Just as domain walls produced in a non-inflationary universe would close it, so would monopoles. It was to dilute the monopole density to an acceptable level that Guth originally introduced inflation. This does not mean that all defects are diluted. There is no difficulty in developing hybrid models in which monopoles are diluted but strings survive [15].

All the above is only the simplest possibilities. With suitable symmetry breaking we can have strings ending in monopoles, domain walls bounded by strings, and so on.

C. The observation of defects

Let us postpone estimating how many defects we might see at a transition until later, and ask the simpler question as to whether we see examples of these defects in real life. All the simple defects permissable in condensed matter systems are seen in the laboratory, as are many more complicated combinations of defects. See articles in [1, 2]

In cosmology the situation is the converse, with no unambiguous sightings of monopoles or strings (vortices). Cosmic strings can be best observed by gravitational lensing, in which no central object is seen. The metric around a cosmic string is conical, with a defect angle $\delta = 8\pi(T_c/M_{Pl})^2 \sim 10^{-5}$. Recently a plausible galaxy string lensing candidate was observed [17], but in the absence of further lensing nearby, the existence of a cosmic string is in doubt. Cosmic strings are, in principle, a potential source of Extra High Energy Cosmic Rays (EHECR) [18]. However, simple mechanisms like loop decay, cusp radiation and string interconnection give a flux many orders of magnitude (> 10) too small. There are many other variants, among which vorton loops and loops of superconducting string are the more promising [19]. In the latter case, the superconducting current increases until the loop disintegrates, producing EHECR.

The most promising source of quark-hadron transitions is the interior of neutron stars [11, 20]. The quark-gluon condensate is highly compressible, compared to nuclear matter. Arguably this is seen in millisecond pulsars. As they spin-up via accretion the core-density decreases and the core turns to hadrons. As the quark-matter spins it inhibits an increase in the pulsar frequency ω because of an increase in the moment of inertia. Detailed calcu-

lation suggests a frequency ($\omega \sim 300Hz$) at which the pulsar distribution should peak, and this is observed experimentally [21].

The vortices and vortons that are produced at the transition provide candidates for the observed glitches in pulsar frequencies ($\delta\omega/\omega \leq 10^{-6}$). The suggestion is that these are understood as a consequence of the interactions between the rigid crust of the neutron star and vortices/vortons in the neutron superfluid interior as they try to move outwards.

VIII. THE DYNAMICS OF DEFECT PRODUCTION

A. Causal Bounds

Because phase transitions take place in a finite time, causality guarantees that correlation lengths remain finite. If the symmetry breaking permits, defects arise so as to mediate the correlated regions with different ground states. They provide an excellent experimental signature for the way in which transitions are implemented.

For condensed matter systems, Zurek[22, 23] suggested that causality alone is sufficient to determine the initial density of defects arising in a continuous transition. In this he paralleled proposals made by Kibble[24] in the context of quantum field theory models of the early universe.

Consider a system with critical temperature T_c, cooled through that temperature so that, if $T(t)$ is the temperature at time t, then $T(0) = T_c$. $\dot{T}(0) = T_c/\tau_Q$ defines the quench time τ_Q. One formulation (see [23]) of the Zurek/Kibble causal bound is the following: Suppose, at time t, that $c(t) = c(T(t))$ is the maximum speed at which the order parameter can respond to a change in the environment. Then, at time t the causal horizon has diameter

$$h(t) = 2\int_0^t ds\, c(s). \tag{10}$$

Let $\xi_{ad}(t) = \xi_{ad}(T(t))$ be the adiabatic healing length (correlation length), diverging at time $t = 0$ ($T = T_c$). The first time that defects can appear is at time $\bar{t}$, when the causal horizon becomes big enough to enclose one of them, $\xi_{ad}(\bar{t}) = h(\bar{t})$.

As a direct consequence, $\bar{t}$ has the form $\bar{t} = \tau_Q^{1-\gamma}\tau_0^{\gamma}$. The scaling exponent γ depends upon the system. For dissipative condensed matter, with critical slowing down ($c(0) = 0$), $\tau_0 \ll \tau_Q$ is the relaxation time of the longest wavelength modes, for QFT (with c a constant) the time for light to traverse a Compton wavelength. As a result, $\tau_Q \gg \bar{t} \gg \tau_0$. If $\xi_{ad}(t) \sim \xi_0(t/\tau_Q)^{-\nu}$ for $t \sim 0$, where $\xi_0 = O(\xi_{ad}(T = 0))$, then the initial domain size and defect separation is predicted to be

$$\bar{\xi} \sim \xi_{ad}(\bar{t}) = \xi_0\left(\frac{\tau_Q}{\tau_0}\right)^{\sigma} \gg \xi_0, \tag{11}$$

where $\sigma = \gamma\nu$. This is very large on the scale of cold defects. We term σ the Zurek-Kibble (ZK) characteristic index. In the mean-field approximation $\sigma = 1/4$ typically for condensed matter, and $\sigma = 1/3$ for QFT.

We see that the same arguments apply to crossovers [25], provided they are weak enough that the correlation length is larger than the causal horizon at time $\bar{t}$, even though it does not diverge. We assume this to be the case.

B. Unstable modes

In the several years since these simple bounds were first proposed we have acquired a much better understanding of the way in which transitions occur. These does not mean that these bounds have lost their relevance, but that they need to be qualified.

Any dynamical equations for the onset of a continuous transition will embody causality, by definition. However, the transition cannot be said to have happened before the order parameter has achieved its equilibrium value $|\phi|^2 = \eta^2$, (in our notation for potentials). If $\langle...\rangle_t$ denotes ensemble averaging at time t then a lower bound on the first time from which we can start counting defects is $t = t^*$, for which $\langle|\phi|^2\rangle_t = \eta^2$.

For many condensed matter systems and for quantum fields, the way in which $\langle|\phi|^2\rangle_t$ builds up to its final value is by the growth of the amplitudes of the unstable long-wavelength modes, which are unstable because of the upturned parabolic free energy density at initial times. The time t^* is, crudely, the time for these modes to roll from the top of the hill to the groundstates at the bottom, related to the spinodal time.

The equations that control this ordering through instabilities differ for different systems.

C. Dissipative systems:

In the vicinity of T_c, dissipative systems are often well approximated by the time-dependent Ginzburg-Landau equations (TDGL). For a global theory these are Langevin equations of the form

$$\frac{1}{\Gamma}\frac{\partial\phi_a}{\partial t} = -\frac{\delta F}{\delta\phi_a} + \eta_a, \tag{12}$$

where η_a is Gaussian thermal noise, satisfying $\langle\eta_a(\mathbf{x},t)\eta_b(\mathbf{y},t')\rangle = 2\delta_{ab}T(t)\Gamma\delta(\mathbf{x}-\mathbf{y})\delta(t-t')$. This is a crude approximation for ^4He, and a simplified form of a realistic description of ^3He [7], and an important part of the description of low-T_c superconductors.

This seems a very different picture from that originally proposed by Zurek. A priori, t^* is not related directly to the causal $\bar{t}$, but it is not difficult to see why they might be comparable. Unstable modes grow exponentially fast. As long as dimensional analysis makes $\bar{t}$ the natural unit in which to measure time, any exponentially

growing term will achieve values that are not exponentially large at times $t = O(\bar{t})$. This can be checked explicitly for fast quenches on retaining only the quadratic part of F, assuming that growth in $\langle |\phi|^2 \rangle_t$ is then checked by the (linearised) back-reaction. This is justified by seeing that the growth in long wavelength modes is largely completed in the linear regime, when the field components ϕ_a are effectively independent [26, 27].

D. Relativistic QFT:

For QFT the situation is rather different. In the previous section, instead of working with the TDLG equation, we could have worked with the equivalent Fokker-Planck equation for the probability $p_t[\Phi]$ that, at time $t > 0$, the measurement of ϕ will give the function $\Phi(\mathbf{x})$. When solving the dynamical equations for a hot quantum field it is convenient to work with probabilities from the start.

The probability $p_t[\Phi]$ that, at time t, the measurement of ϕ will give the value Φ is $p_t[\Phi] = |\Psi|^2$, where Ψ_0 is the state-functional with the specified initial condition (e.g. Boltzmann distributed). In the language of path integrals, $p_t[\Phi]$ can be written as an integral for fields with increasing time (Ψ) followed by an integral for fields with decreasing time (Ψ^*). The time contours can be joined to give a closed time-path and non-equilibrium calculations in QFT are termed closed time-path calculations.

In practice, there is no need to calculate $p_t[\Phi]$ directly. If $\langle |\Phi(\vec{x})|^2 \rangle_t$ measures the growth of field modes as an ensemble average with respect to $p_t[\Phi]$, then

$$\langle |\Phi(\vec{x})|^2 \rangle_t = \langle |\phi(\vec{x}, t)|^2 \rangle$$

for Wightman fields $\phi(\vec{x}, t)$, subject to the thermal boundary conditions at the initial time before the transition is implemented. Mode analysis is all but impossible outside self-consistent linearisation of the back-reaction [28]. Within the self-consistent linear regime (which is the best that can be done numerically) the mode equations that determine $\langle |\phi|^2 \rangle_t$ are now the classical second-order equations

$$\frac{\partial^2 \phi_a}{\partial t^2} = -\frac{\delta F}{\delta \phi_a}. \tag{13}$$

Yet again, this looks a rather different picture from that originally proposed by Kibble, in which the system froze in before the transition was effected. Nonetheless, although t^* is not related directly to the causal $\bar{t}$, they are again comparable, essentially because unstable modes grow exponentially fast, and $\bar{t}$ has the correct engineering dimensions.

There is one major difference between condensed matter systems and the early universe that is not addressed in equations like (13). This is that, whereas the effective Ginzburg-Landau field theories for condensed matter systems are complete, the early universe contains many (probably most) fields about which we know nothing, and

which are ignored in (13). Only for QCD to we have a full tally of the relevant degrees of freedom of the system. The effect of this environment is to make the quantum field theory behave classically (show decoherence) [29, 30]. The easiest way to show this is by seeing how the density matrix (whose diagonal elements are $p_t[\Phi]$) becomes diagonal as a consequence of the growth of the unstable modes. In parallel, the master equation for the Wigner functional plays the role of a Fokker-Planck equation, whose Langevin counterpart for the semiclassical field is a variant of (12), including multiplicative noise. However, none of this interferes strongly with defining t^* from (13).

What is important is that, by the time that defects are produced they behave like classical entities [30].

E. Other systems

Not all systems behave as the above. Bose-Einstein condensates satisfy the Gross-Pitaevski equation [10]

$$i\hbar \frac{\partial \phi_a}{\partial t} = \frac{\delta F}{\delta \phi_a}. \tag{14}$$

Although causality still leads to the Zurek bounds, transitions do not evolve because of the exponential growth of unstable modes. A similar situation exists for those transitions that are driven by changes in chemical potentials (as in high-T_c superconductors).

F. Additional mechanisms for gauge theories

All the mechanisms above are well-suited for global symmetry-breaking, but for local symmetry-breaking in the presence of gauge fields there is an additional mechanism for the production of defects. This mechanism, observed by Hindmarsh and Rajantie [31] is discussed in detail in Arttu Rajantie's contribution to these proceedings, and will only be considered briefly here.

For the most relevant case of superconductors (both high and low temperature) the idea is simple, even if the execution is difficult. Above the critical temperature there are thermal fluctuations in the electromagnetic field. As the system cools through T_c the short wavelength modes of the field stay in equilibrium, whereas long wavelength modes drop out of equilibrium and freeze in. These frozen modes are the source of correlated flux that will be measured along with the flux of conventional Abrikosov vortices produced from the causal arguments above. The effect is not simply additive but will, in general, lead to a greater variance in the spontaneously produced flux passing through a surface.

G. Defects as zeroes

So far we have only been discussing the time t^* it takes for the transition to complete itself, and comparing this to the causal time $\bar{t}$. The important step is to derive the defect separation ξ^* at the time of defect production and compare it to $\bar{\xi}$ of (11).

To do this, we identify simple defects, with normal (false vacuum) cores, by the zeroes of the order parameter fields. At early times, when field fluctuations are strong, field zeroes do not have the energy profiles to be identified with defects, whose masses, tensions, etc, are $O(\eta)$ non-perturbatively large. However, by t^*, the energy profiles are qualitatively correct, and counting zeroes provides a reliable estimate of defect numbers. [In the case of QFT, this requires that defects can be considered classically.]

In the linear regime the separation of zeroes is given [32] simply in turns of the derivatives of the two-point correlation function $\langle \phi_a(\vec{x}, t) \phi_b(\vec{0}, t) \rangle$. In this approximation we find that, just as $t^* \approx \bar{t}$ (up to logarithmic corrections), then so is $\xi^* \approx \bar{\xi}$.

We have observed that many systems have defects with complicated cores. Simple numerical simulations show that, provided the non-normal components of the cores are not too large, the scaling laws of the causal analysis are preserved [33].

There is a further important comment on thermal fluctuations. Initially [34], it was thought that domain size after a transition was determined by the correlation length at the Ginzburg temperature T_c. In the causal arguments above the Ginzburg regime plays no role, but this reappears in the analysis of unstable modes, where thermal fluctuations on small scales can make the production of defects difficult [35] if the quench is too slow.

IX. EXPERIMENTAL CONFIRMATION

To date, several condensed matter experiments had been performed to test (11):

- **Superfluid** $^3He - B$[36, 37]. The two experiments[36, 37] on $^3He - B$ rely on the fact that, when superfluid $^3He - B$ is bombarded with slow neutrons, $n +^3 He \rightarrow p +^3 H + 760\text{keV}$. The energy released in such a collision leads to a hot spot in the superfluid, with temperature $T > T_c$, which when cooled by its environment, leaves behind a tangle of vortices (the topological defects in this system). τ_Q is fixed by the nuclear process that breaks up the ^3He atom. With only a single data point conflating both normalisation and σ it is not possible to confirm the predicted value $\sigma = 1/4$. However, both experiments are highly compatible with (11).

- **Superfluid** ^4He[38, 39]. In principle, the two ^4He experiments[38, 39], which use a pressure quench, allow for a more complete test. Yet again, vortices are the relevant defects. In practice, the most reliable experiment[39] sees no vortices. This is not necessarily a sign of contradiction in that it has been suggested[35] that the vortices decay too fast to be seen in this case. This is irrespective[40] of whether high thermal fluctuations within the wide Ginzburg regime of ^4He would lead to somewhat different predictions. In this context, the vortices seen in an earlier ^4He experiment[38] were most likely an artefact of the experimental setup.

- **High temperature superconductors (HTSC)**[41, 42]. The two experiments[41], on HTSC measure the total flux of Abrikosov vortices through a surface. The vortex separation of (11) can be converted into a prediction for the variance. In the first experiment no flux was seen, despite the phase separation that leads to the result being demonstrated elsewhere[43]. However, on increasing the quench rate by several orders of magnitude spontaneous flux was produced [42], but with low efficiency. The observed flux is compatible with the Zurek prediction of (11), once additional assumptions for converting total flux into net flux are taken into account. Although the prediction (11) has not taken gauge fields into account, the effect of electromagnetic field fluctuations is expected to be small[44].

- **Annular Josephson Tunnel Junctions**. The defects of a lineaer JTJ are fluxons, the sine-Gordon model kinks in the field $\phi = \phi_1 - \phi_2$, the difference in the phases of the complex order parameter fields in the separate superconductors. Again two experiments have been performed (one currently in progress, with data as yet unanalysed). The first experiment [45] was one in which, by counting fluxons on varying quench time τ_Q, it was possible to compare σ with its theoretical value (as well as confirming overall scale). The secondary mechanism of [31] is not relevant here. Agreement with both exponent and magnitude are good, although there is scatter, with $\sigma = 0.27 \pm 0.05$, in comparison to the theoretical value of $\sigma = 0.25$. The new experiment currently being performed will permit quench times four orders of magnitude faster than before, and thereby an order of magnitude more fluxons.

- **Other direct experiments**. Two subsequent experiments have permitted varying quench rates and so an estimate for σ. The first [46] involves the Bénard-Marangoni conduction-convection transition, in which a homogeneous conduction state is broken into an hexagonal array of convection lines on heating. The defects here are not associated with the line zeroes of an order parameter field, and the viscosity-dependent σ does not match the

ZK prediction, possibly for that reason. The second experiment[47] is carried out in a non-linear optical system, with complex beam-phase the order parameter. Increasing the light intensity (the control parameter in this case) leads to pattern formation (defects) at a critical value. The predicted $\sigma = 1/4$ is recovered to good accuracy as $\sigma_{exp} = 0.25 \pm 0.02$.

- **Related experiments**. Related experiments include measurements of defect production in continuous transitions of planar liquid crystals [9, 48], measurements of flux production in dilute Josephson Junction annuli [49], and measurements at first order transitions in radioactive low-T_c superconductors [50]. The first provides a check on the relationships between net and gross topological charge, the second on the role of electromagnetic field fluctuations at a transition [44], relevant for the high-T_c experiments. At the moment the implications of the third are unclear, given the different nature of first order transitions.

X. CONCLUSIONS

We have seen that the patterns of symmetry breaking in condensed matter systems and the early universe are sufficiently rich and sufficiently similar to pursue the comparison.

In almost all cases the transitions are signalled by the production of defects, usually topological vortices, vortons or monopoles. The observation of defects is therefore a demonstration that the transitions have taken place. Defect production in the early universe is complicated by the assumed inflationary period, and there are only tantalising glimpses of possible candidates. The situation is very different for condensed matter systems, where defects are readily observed.

In this latter case we can do more, and examine the details of defect production. A comparison of the experimental results with the causal constraints on defect production proposed by Zurek shows the strength and limitations of the causal bounds and informs the similar arguments on the role of causality in defect production in the early universe, as originally mooted by Kibble.

[1] *Topological Defects and the Non-Equilibrium Dynamics of Symmetry Breaking Phase Transitions*, Proceedings of the Nato ASI, Les Houches (2000), Y.M. Bunkov and H. Godfrin (eds.), Nato Science Series **C549**, Kluwer Academic Publishers (2001).
[2] *Patterns of Symmetry Breaking*, Proceedings of the COSLAB meeting at Crakow (2002), H. Arodz et al. (eds.), Kluwer Academic Publishers, Dordrecht (2003).
[3] S-C. Zhang, eprint cond-mat/9610140, Science **275**, 1089 (1997)
[4] D.P. Arovas, A.J. Berlinsky, C. Kallin and S-C. Zhang, *Phys. Rev. Lett.* **79**, 2871 (1997).
[5] S. Alama, A.J. Berlinsky, L. Brossard and T.Giorgi, *Phys. Rev.* **B60**, 6901 (1999).
[6] *Exotic Properties of Superfluid ^3He*, G.E. Volovik, World Scientific, Singapore (1992).
[7] Yu. M. Bunkov and O.D. Timofeevskaya, *Phys. Rev. Lett.* **80**, 4927 (1998).
[8] I. Chuang, R. Durrer, N. Turok and B. Yurke, *Science***251**, 1336 (1991)
[9] S. Digal. R.Ray and A.M. Srivastava *Phys. Rev. Lett* **83**, 5030 (1999).
[10] M.A. Metlitski and A.R. Zhitnitsky, cond-mat/0307559 (2003)
[11] K. Rajagopal, Chapter 35 in the Festschrift in honor of B. L. Ioffe, "At the Frontier of Particle Physics / Handbook of QCD", M. Shifman, ed., (World Scientific, Singapore), 2001, hep-ph/0011333.
[12] C. Korthals-Altes, A. Kovner and M. Stephanov, *Phys.Lett.* **B469**, 205 (1999).
[13] K.B.W. Buckley and A.R. Zhitnitsky, *JHEP***208**, 13 (2002)
[14] E. Witten, *Nucl. Phys* **B249**, 557 (1985)
[15] R. Jeannerot, J. Rocher and M. Sakellariadou *Phys.Rev.* **D68** 103514 (2003).
[16] K.B.W. Buckley and A.R. Zhitnitsky, *Phys. Rev.***B67** 174522 (2003).
[17] M. Sazhin et al., *Mon. Not. Roy. Astron. Soc.***343**, 353 (2003), astro-ph/0302547
[18] P. Bhattachargee and G. Sigl, *Phys. Rep.***327**, 109 (2000)
[19] S. Bonazzola and P. Peter, *Astropart. Phys.***7**, 161 (1997)
[20] D. Boyanovsky, H.J. de Vega, M. Simionato *The Early Universe and the Cosmic Microwave Background: Theory and Observations*, N.G. Sanchez and Y.N. Parijskij (eds.), Nato Science Series II **150**, Kluwer Academic Publishers (2003), 65.
[21] N.K. Glendenning and F. Weber, *Astrophys. J.***480L** 111 (1997).
[22] W.H. Zurek, *Nature* **317**, 505 (1985), *Acta Physica Polonica* **B24**, 1301 (1993).
[23] W.H. Zurek, *Physics Reports* **276**, 177 (1996).
[24] T.W.B. Kibble, in *Common Trends in Particle and Condensed Matter Physics*, *Physics Reports* **67**, 183 (1980).
[25] N. Antunes, L.M. Bettencourt, W. H. Zurek *Phys. Rev. Lett* **82**, 2824 (1999).
[26] E. Kavoussanaki, R.J. Rivers and G. Karra, *Condensed Matter Physics* **3**, 133 (2000).
[27] R.J. Rivers, *Journal of Low Temperature Physics* **124**, 41-84, (2001).
[28] D. Boyanovsky, Da-Shin Lee and A. Singh, *Phys. Rev.*

D48, 800 (1993),

D. Boyanovsky, H.J. de Vega and R. Holman, *Phys. Rev.* **D49**, 2769 (1994).

[29] R.J. Rivers, F.C. Lombardo, F.D. Mazzitelli, *Phys. Lett.* **B 523**, 317 (2002)

[30] R.J. Rivers, F.C. Lombardo, F.D. Mazzitelli, *Phys. Lett.* **B 539**, 1 (2002)

[31] M. Hindmarsh and A.Rajantie, *Phys. Rev. Lett.* **85**, 4660 (2000); A. Rajantie, *Journal of Low Temperature Physics* **124**, 5 (2001).

[32] F. Liu and G.F. Mazenko, *Phys. Rev.* **B46**, 5963 (1992).

[33] N. Antunes, P. Gandra, R.J. Rivers and A. Swarup, in preparation.

[34] T.W.B. Kibble, *J. Phys.* **A 9**, 1387 (1976).

[35] G. Karra and R.J. Rivers, *Phys. Rev. Lett.* **81**, 3707 (1998).

[36] C. Bauerle *et al.*, *Nature* **382**, 332 (1996).

[37] V.M.H. Ruutu *et al.*, *Nature* **382**, 334 (1996).

[38] P.C. Hendry *et al*, *Nature* **368**, 315 (1994).

[39] M.E. Dodd *et al.*, *Phys. Rev. Lett.* **81**, 3703 (1998), *J. Low Temp. Physics* **15**, 89 (1999).

[40] R.J. Rivers, *Phys. Rev. Lett* **84**, 1248 (2000).

[41] R. Carmi and E. Polturak, *Phys. Rev.* **B 60**, 7595 (1999).

[42] A. Maniv, E. Polturak and G. Koren, eprint cond-mat/0304359.

[43] R. Carmi, E. Polturak, and G. Koren, *Phys. Rev. Letts.* **84**, 4966 (2000).

[44] T.W.B. Kibble and A. Rajantie, eprint cond-mat/0306633.

[45] R. Monaco, J. Mygind and R.J. Rivers *Phys. Rev. Lett.* **89**, 080603 (2002); *Phys. Rev.* **B 67**, 104506 (2003).

[46] S. Casado, W. González-Viñas, H. Mancini and S. Boccaletti, *Phys. Rev.* **E63**, 057301 (2001).

[47] S. Ducci, P.L. Ramazza, W. González-Viñas, and F.T. Arecchi, *Phys. Rev. Lett.* **83**, 5210 (1999).

[48] R.J. Rivers and A. Swarup, cond-mat/0312082.

[49] J.R. Kirtley, C.C. Tsuei and F. Tafuri, arXiv:cond-mat/0304359

[50] J. Pramos, O. Bertolami, T.A. Girard, P. Valko, *Phys.Rev.* **B67** 134511 (2003).